Minimal Contagious Sets in Innovation Diffusion Networks*

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Abstract

We investigate the minimal seed size required for successful innovation adoption in networks of growing size, such as scale-free networks, which are widely used to model real-world social networks. We use the bootstrap percolation dynamics model, where agents adopt the innovation if a certain fraction of their neighbors has already adopted it. Our main contribution is to provide an upper and lower bound on the size of the seed set and to prove that in growing networks, the minimal seed required to reach the entire network cannot be sub-linear in the network size. We demonstrate this result through simulations. Our study contributes to a better understanding of innovation diffusion in scale-free networks and offers potential applications in marketing, public health, and social policy. Our results provide a theoretical foundation for future research to investigate optimal seeding strategies in more complex network structures and dynamics.

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1 Introduction

Innovation diffusion is a fundamental process that enables the spread of new technologies, products, and ideas. Typically, the process starts with a small *seed* of people known also as early adopters or innovators. These individuals are less influenced by the opinions and actions of others within their network, therefore are often the first to see the potential of new technology and are motivated to adopt it even if it is unproven or untested. This early adoption creates local positive externalities, which encourage other agents to adopt the innovation. One popular diffusion model is the bootstrap percolation dynamics, which originated in physics and has been since applied to a wide range of fields, from epidemiology to marketing (see the related literature). In this model, agents adopt (for good) the innovation if a certain threshold of their neighbors has already adopted it (the threshold is either a fixed number of neighbors or, as in our model, a proportion of them). The process continues, until reaching a steady state. If in this steady state, the innovation was adopted by the entire society, we say that the seed is *contagious* or *percolates*.

This can be the case when adopting a new technology or product is costly or requires effort. For example, switching a cellular provider is costly, but can have internal benefits (better customer service, better reception) and local positive externalities in case calls and texts to users using another provider are more expensive than to users using the same one. Thus, even an agent reluctant to switch a provider at first might do so, once enough of his contacts switch a provider, to reduce his bills. This effect, along with social learning regarding the quality of the new provider, was shown to exist in Hu et al. (2019) and typically characterizes high-tech products. In other cases, the adoption process is driven by "word-of-mouth" marketing (Narayan et al., 2011; Confente, 2015; Iyengar et al., 2011) or their combination.

An important marketing question is how many early adopters are needed in order for an innovation to be adopted by everyone, assuming that one has the ability to "target" individuals and determine the seed. In other words, what is the size of the smallest contagious seed? The answer depends strongly on the network structure and was answered in the literature for some particular cases. For example, for a star-shaped network, one innovator (the central) is always enough, regardless of the number of agents. Chalupa et al. (1979) (and the subsequent physics-related literature) answered this question mainly for grids, Chang and Lyuu (2010) and Ackerman et al. (2010) focused on networks with a fixed size and different local thresholds, whereas Angel and Kolesnik (2018) and Guggiola and Semerjian (2015) studied the minimal contagious seed for random graphs. Note that unlike other models of opinion dynamics and reaching a consensus (such as the ones presented in Rosenberg et al. (2009) and Bikhchandani et al. (2021)), we do not assume that there is an underlying true state to be learned or a "correct" action to be taken. The only driving force in our model is the externalities due to peer pressure.

While some of the previous studies can be applied to real-world social networks, these frameworks do not take into account their unique structure. Social networks often exhibit a dynamic nature, in which new agents are constantly added to the network through some attachment mechanism. For example, preferential attachment (Barabási and Albert, 1999; Albert and Barabási, 2002). leads to a particular structure, known as a scale-free topology, in which the degree distribution converges to a power-law distribution (although there is an ongoing debate regarding the exact distribution, see Broido and Clauset (2019); Stumpf and Porter (2012) and Holme (2019)). In addition to social networks, this degree distribution approximates many other real-life networks, such as power grids, computer networks, and citations of research papers (see Figure 24 and Table 1 in Dorogovtsev and Mendes (2002) for details).

In this paper, we exploit the particular structure of growing networks with converging degree distribution (such as the ones presented in Krapivsky and Krioukov (2008); Barabási and Albert (1999); Falkenberg et al. (2020), and Choromański et al. (2013). See also the review paper by Dorogovtsev and Mendes (2002)) and provide a lower and upper bound on the size of a contagious seed for bootstrap percolation dynamics with any majority threshold (similarly to Amini et al. (2013), who assume a fixed threshold). Our main result (Theorem 2) is that the size of the contagious seed can be bounded from below by a linear function of the size of the population, where the linearity coefficient is determined by the degree distribution. We conclude that outside of knife-edge cases like the star-shaped network, it is impossible to get contagion without seeding a significant fraction of the population.

This impossibility result shows the resilience of networks to being overtaken by bootstrap percolation and highlights the importance of a sufficiently large seed for successful innovation diffusion. Our findings have implications for industries and policymakers looking to understand how to accelerate technology adoption and diffusion in real-world social networks, and in general, in planning marketing campaigns (see also Akbarpour et al. (2020) on the added value of just a few more seeds). For example, our results suggest that targeted seeding strategies, such as identifying and leveraging well-connected early adopters (Demange, 2017, 2018), may be less effective than mass marketing campaigns regarding seed size. Furthermore, our work could lay the path to the development of more accurate models of innovation diffusion that take into account the underlying network structure. Overall, our study offers a better understanding of the dynamics of innovation diffusion in scale-free networks, with potential applications in fields such as marketing, public health, and social policy.

Structure of the paper. This introduction is followed by a discussion of the related literature. In Section 2 we describe the model and the main results. The proofs are presented in Sections 3, 4, and 5. In Section 6 we study tightness of the bound in general and for scale-free networks through computer simulations, followed by a comparison of majority and minority dynamics (Section 6.3). We show that there is a discontinuity in the lower bound when the required fraction of neighbors needed for activation changes from slightly less than 50% to 50%. The discussion in Section 7 concludes the paper.

1.1 Related Literature

Bootstrap percolation was first presented in Chalupa et al. (1979) in the context of solid-state physics and was studied in different fields such as mathematics, computer science, marketing, communication, and epidemiology (see for example Drakopoulos et al. (2014); Freund et al. (2018); Guggiola and Semerjian (2015); Manshadi et al. (2020); Garbe et al. (2018); Berger (2001) and the references within, especially those in Angel and Kolesnik (2018)). In all these studies, a seed of activated vertices influences their neighbors and possibly activates them too, until a steady state (in some sense) is reached. The main differences, that emerge from the particular application of the model are regarding the selection of the seed set (is it randomly chosen or not), the structure of the graph (Erdős–Rényi, regular, scale-free, and so on), the law of infection (most commonly: is a vertex activated because an absolute number of its neighbors is active, or because a certain fraction of its neighbors is active), the percolation objective (activate the entire network or just a fraction of it) and the persistence of the seed (can active vertices become inactive or not).

In the economic context, the model was applied to study the collapse cascade financial institutes (Kempe et al., 2003; Elliott et al., 2014; Amini and Minca, 2016; Amini et al., 2016; Chen, 2009; Gai and Kapadia, 2010; Demange, 2018), risk in supply-chains (Li et al., 2020), and "word-of-mouth" marketing theoretically (Goel et al. (2016); Freund et al. (2018); Manshadi et al. (2020) and the references within) and empirically (Hu et al., 2019; Iyengar et al., 2011). When the vertices represent individuals, such as in marketing or innovation diffusion, a plausible assumption is that the structure of the network is a scale-free network (Albert and Barabási, 2002), which naturally emerges when new individuals are added to the network and connected to existing ones through the preferential attachment model (another possibility is that connections depend also on their real-life distance, see Gao et al. (2015)).

In this paper, we study propagation through scale-free networks (more generally: $\mathbb{W}_1$-converging networks, see Definition 1) via the bootstrap percolation dynamics with a fractional threshold and the objective of reaching the entire population. We show that $\mathbb{W}_1$-converging networks are decentralized enough, so a small seed is not enough to activate the entire network. We also provide an upper bound for growing networks through some attachment process (Definition 2) which improves previously known bounds and, in addition, present a novel characterization of the contagious seed set. For scale-free networks, our bound shows that at least 1% of the network must be chosen as the initial seed (and any set which is sub-linear in the size of the network is not enough), and at most $25\% - 50\%$ of the network is needed, depending on the exact parameters of the distribution. Our simulation shows that the true size of the seed is roughly 10% of the network.

Previously, bootstrap percolation on scale-free networks was studied mainly with an absolute threshold (say, three, as in Amini and Fountoulakis (2012)). There it is shown that the seed set can be sub-linear (see also Freund et al. (2018)) when it is optimally chosen. On the contrary, when the seed is randomly chosen, a sublinear portion of the network is no longer enough (Amini et al., 2013)).

2 Model and Results

We consider innovation diffusion in a network using the bootstrap percolation dynamics with threshold $\rho \in [0,1)$. A social network is a finite undirected simple graph $G = (V,E)$, where V is the set of n agents (vertices) and $E \subseteq V \times V$ is the set of edges. The set of neighbors of agent v is denoted N_v and the degree is $d_v = |N_v|$. The cumulative distribution function (CDF) of the degrees $F\colon \mathbb{Z}_+ \to [0,1]$ is defined to be $F(d) = \frac{1}{n}|\{v : d_v \le d\}|$. Finally, $\Delta_1(\mathbb{N})$ is defined to be the space of all finite-expectation probability measures on $\mathbb{N} = \{1,2,\ldots\}$, so $F \in \Delta_1(\mathbb{N})$.

A certain innovation is introduced and an agent can adopt it (then he is *activated* or *infected* (Garbe et al., 2018)) or not. At time $t = 0$, a set $A_0 \subset V$ of agents is activated. These agents are called *seeds* and A_0 is called the *seed set*. Those seeds stick to the innovation thereafter regardless of what others do. The other agents choose between the two alternatives every day ($t = 1,2,\ldots$). An inactive agent adopts the new technology if and only if he sees that strictly more than a fraction ρ of his neighbors were activated. In this case, we say that the agent sees that a ρ-majority of its neighbors were activated. Formally, the sets $A_0 \subset A_1 \subset \cdots$ of active agents at times $t = 0,1,\ldots$ are defined[1] recursively by

$$A_{t+1} = A_t \cup \{v \in V : |N_v \cap A_t| > \rho d_v\}.$$

Let $A_\infty := \langle A_0, G\rangle_\rho := \bigcup_{t=0}^{\infty} A_t$. When the graph G is finite there exists t_0 such that $A_t = A_\infty$, for all $t \ge t_0$. If $A_\infty = V$, A_0 is said to be *ρ-contagious* (or *percolates*). When ρ is clear from the context, we may omit it saying just *contagious*. Our goal is to characterize the size of the smallest contagious seed, denoted by $h_\rho(G)$.

Figure 1 depicts the dynamics for $\rho = 0.5$, where the white vertices are activate (A_t) and the black are inactive ($\bar{A}_t := V \setminus A_t$). It is easy to verify that $h_{0.5}(G) = 2$ in this example.

Our main results are upper and lower bounds on $h_\rho(G)$ in terms of the distribution of the degrees of the agents of G. The first main result is a lower bound that applies to any finite network. Our result relies on the entire distribution of the degrees of the network, in contrast to existing results (such as Chang and Lyuu (2010)) that rely only on some statistics of that distribution (such as the maximal degree).

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[1] In principle, we could allow active agents to become inactive again by postulating $v \in A_{t+1}$ if and only if $v \in A_0$, or $|N_v \cap A_t| > \rho d_v$, or $|N_v \cap A_t| = \rho d_v$ and $v \in A_t$. The two definitions are equivalent since the seeds never become inactive, and by induction on t, any agent that becomes active at time t remains active forever.

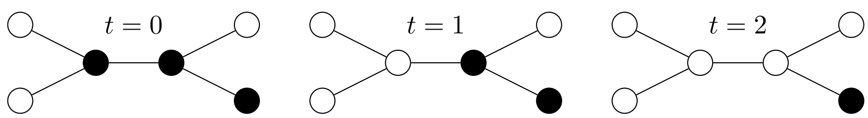

Figure 1: An example of the dynamics with a seed of three (white) vertices for $\rho = 0.5$. At time $t = 3$, the last black vertex activates so $A_3 = V$, and therefore, the seed is contagious.

Theorem 1. *There exists* $\alpha\colon \Delta_1(\mathbb{N}) \to (0, 1]$ *such that for every* $\rho \geq \frac{1}{2}$, $n \in \mathbb{N}$, *and network* G *with* n *agents and degree distribution* F_G,

$$h_\rho(G) \geq \alpha(F_G)n.$$

The function α is given a closed-form expression in Section 3. Our α works for $\rho = \frac{1}{2}$, and therefore, for $\rho > \frac{1}{2}$ as well. It would be interesting to find larger lower bounds α_ρ that depend on ρ and also bounds that take into account agent-dependent thresholds, i.e., different $\rho_v \geq 0.5$ for each $v \in V$, as in Ackerman et al. (2010); Chang and Lyuu (2010).

Despite its appearance, the bound in Theorem 1 is not necessarily linear in n, since $\alpha(F_G)$ may tend to 0 for larger and larger networks. For example, for the complete network, K_n, we have $\alpha(F_{K_n}) = \frac{1}{n+1}$, so our bound is vacuous.

Our second main result, Theorem 2, provides conditions on a sequence of networks $\{G_n\}_{n=1}^{\infty}$ under which the sequence of coefficients $\alpha(F_{G_n})$ converges, which ensures that the lower bound on $h_\rho(G_n)$ *is* linear in n. The following definition sets these conditions.

Definition 1. *A family of networks* $(G_n = (V_n, E_n))_{n\in\mathbb{N}}$ *of sizes* $|V_n| = n$ *with* F_n *being the CDF of the degrees of* G_n *is* $\mathbb{W}_1$-converging *if there exists a distribution on* $\mathbb{N}$ *with CDF* F *such that:*

1. *F is the pointwise limit of F_n, i.e.* $F_n(d) \xrightarrow[n\to\infty]{} F(d)$, *for every* $d \in \mathbb{N}$.
2. *The average degree of F_n converges to the average degree of F, i.e.* $\mathbb{E}(F_n) \xrightarrow[n\to\infty]{} \mathbb{E}(F)$, *where* $\mathbb{E}(F) := \int x \,\mathrm{d}F(x)$.

Theorem 2. *Let* $(G_n)_{n\in\mathbb{N}}$ *be an* $\mathbb{W}_1$*-converging sequence of networks with* F_n *being the CDF of the degrees of* G_n *and* F *their limit distribution. If* $\mathbb{E}(F) < \infty$ *then* $h_\rho(G_n)$ *is linear in* n*, that is, there exists* $\alpha > 0$ *such that* $h_\rho(G_n) \geq \alpha n$*, for all* $\rho \geq 0.5$.

Definition 1 is equivalent to saying that $F_n \to F$ in L_1, or that $dF_n \to dF$ in the 1-Wasserstein metric[2]. In fact, the proof of Theorem 2 is through showing that $\alpha(F)$ is continuous w.r.t. the 1-Wasserstein metric.

The finiteness of $\mathbb{E}(F)$ is essential for the result to hold. Consider a process of adding agents to a network, such that the degree distribution converges to

[2] See Theorem 6.9 on Page 108 in Villani et al. (2009).

F with $\mathbb{E}(F) = \infty$. Very informally, this implies that during this process, very often we encounter agents with high degrees. These agents are influential, and including them in the seed reduces its size significantly.

In addition, note that the pointwise convergence of $F_n \to F$ is not enough, and as required in Definition 1, the average degree must converge too: $\mathbb{E}(F_n) \to \mathbb{E}(F)$. For example, in a star-shaped network with n agents, the limit of F_n is $F := \mathbf{1}_{[1,\infty)}$ and $\mathbb{E}(F) = 1$, while $\mathbb{E}(F_n) = \frac{2n-1}{n} \to 2$. Indeed, in this case, $h_\rho(G_n) = 1$ which is not linear in n.

An interesting class of $\mathbb{W}_1$-converging networks is the class of preferential attachment networks (Albert and Barabási, 2002; Dorogovtsev and Mendes, 2002). These networks have the feature that agents are constantly added to the network and connected to existing agents according to some rules. In many of the leading models of preferential attachment networks, the construction starts with a certain network of size m_0 and then each new agent is connected to a fixed number $m \leq m_0$ of existing ones. We call networks with the latter property (m_0, m)*-attachment networks*.

Definition 2. *An* (m_0, m)*-attachment network is a network of size at least* m_0 *whose agents can be ordered* $v_1, v_2, \dots$ *such that for every* $l > m_0$ *the degree of* v_l *in the sub-network induced by* $\{v_1, \dots, v_l\}$ *is equal to* m.

For example, in the model of Barabási and Albert (1999), each new agent is connected to an existing one with probability proportional to the latter's degree. The Barabási-Albert Model results in a network where the limiting probability density function (PDF) of the degrees is proportional to d^{-3}. A generalization of this attachment rule yields more general scale-free networks, networks where there exists $\gamma > 2$ such that the limiting PDF is asymptotically proportional to $d^{-\gamma}$, for large d. See, for example, Dorogovtsev and Mendes (2002) who also identify the scale-free property and the value of γ in several real-life networks.

Theorem 3 provides an upper bound on $h_\rho(G)$. This upper bound applies only to attachment networks. We show that for attachment networks, both of the following sets of agents are contagious: the m_0 initial agents plus the set of all agents of degree either at least m/ρ or at most $m/(1-\rho)$.

Theorem 3. *Fix* $\rho \geq 0$*. For any* (m_0, m)*-attachment network* G *with* n *agents and degree CDF* F*, we have*

$$h_\rho(G) \leq m_0 + \min\{1 - F(\lceil \tfrac{m}{\rho} - 1 \rceil), F(\tfrac{m}{1-\rho})\}n. \tag{1}$$

The proof of Theorem 3 utilizes an idea from Morris (2000) which allows an alternative characterizes $h_\rho(G)$. The characterization relates contagious sets to partial orderings of the agents[3] (Proposition 1).

Theorem 3 implies that if $G_1 \subset G_2 \subset \cdots$ is a family of networks generated through a (m_0, m)-attachment process with limiting degree distribution

[3] The partial ordering that corresponds to the smallest contagious set is an object of interest by itself. It can be thought of as an ordinal measure of centrality. Further investigating this point is an interesting direction for future research.

proportional to d^{-3} (as in Barabási and Albert (1999)), then

$$\limsup_{m\to\infty}\limsup_{n\to\infty} h_\rho(G_n)/n \le \rho^2.$$

In the case of $\rho = \frac{1}{2}$, this upper bound is smaller than the general upper bound $h_{\frac{1}{2}}(G) \le \lceil |V(G)|/2\rceil$, which is attained, e.g., by the complete and the line graphs (see Ackerman et al. (2010)).

3 Proof of Theorem 1

Our proof relies on the observation that as the innovation spreads in the network, the number of "disagreements" between neighboring agents decreases. Formally, the *boundary* (also known as *the set of discord*) of a set of agents $A \subset V$ is defined as the set of edges connecting agents in A with agents in $\bar{A}$, denoted $\partial A = \{(v_1, v_2) \in E | v_1 \in A, v_2 \in \bar{A}\}$.

At each time period $t \ge 0$, each agent in $A_{t+1} \setminus A_t$ has more neighbors in A_t than outside of it, so when it is activated, the size of the boundary strictly decreases. More precisely, an agent v is activated if and only if x of its neighbors are active, with $x > \rho d_v$ (and since x must be an integer, $x \ge \lfloor \rho d_v \rfloor + 1$). The net change in the size of the boundary due to its activation is $\Delta_v := (d_v - x) - x \le d_v - 2\lfloor \rho d_v \rfloor - 2 \le -1$. It follows that if A is ρ-contagious, then

$$|\partial A| \ge \sum_{v \notin A} (2 + 2\lfloor \rho d_v \rfloor - d_v) \ge n - |A|. \tag{2}$$

Since $h_\rho(G)$ is non-decreasing in ρ, in the rest of the proof we assume w.l.o.g, that $\rho = \frac{1}{2}$. Let $G = (V, E)$ be a network with n agents and degree distribution CDF $F\colon \mathbb{Z}_+ \to [0,1]$. We denote by $\hat{F}\colon \mathbb{R}_+ \to [0,1]$ the piecewise linear continuation of F to $\mathbb{R}_+$ defined by

$$\hat{F}(x) := (1 - (x - \lfloor x \rfloor))F(\lfloor x \rfloor) + (x - \lfloor x \rfloor)F(\lceil x \rceil). \tag{3}$$

For $A \subset V$, let $d(A)$ be the upper $\frac{1}{n}|A|$-quantile $\hat{F}$ defined as

$$d(A) := \min\{x : 1 - \hat{F}(x) = \tfrac{1}{n}|A|\}.$$

Rename the agents in non-increasing order of degrees so, $d_{v_1} \ge \cdots \ge d_{v_n}$. Note that

$$\tfrac{1}{n}\sum_{i=1}^{|A|} d_{v_i} = \int_{d(A)}^{\infty} \lceil x \rceil \,\mathrm{d}\hat{F}(x).$$

Therefore,

$$\tfrac{1}{n}|\partial A| \le \frac{1}{n}\sum_{v \in A} d_v \le \frac{1}{n}\sum_{i=1}^{|A|} d_{v_i} = \int_{d(A)}^{\infty} \lceil x \rceil \,\mathrm{d}\hat{F}(x).$$

Suppose that A is contagious, then, by Eq. (2) and since $\frac{1}{n}|A| = 1 - \hat{F}(d(A))$,

$$\tfrac{1}{n}|\partial A| \geq \hat{F}(d(A)).$$

By continuity, there exists a d_* such that

$$\hat{F}(d_*) = \int_{d_*}^{\infty} \lceil x \rceil \, \mathrm{d}\hat{F}(x). \tag{4}$$

Moreover, by monotonicity, $\hat{F}(d_*)$ is uniquely defined. Defining $\alpha(F) := 1 - \hat{F}(d_*)$ completes the proof of Theorem 1.

4 Proof of Theorem 2

Let $\{G_n\}_{n=1}^{\infty}$ be a $\mathbb{W}_1$-converging sequence of networks, and let $\{F_n\}_{n=1}^{\infty}$ be their respective degree CDF and F the limiting CDF. By Theorem 1 and since $\alpha(F) > 0$, it is sufficient to show that $\lim_{n\to\infty} \alpha(F_n) = \alpha(F)$.

Let $\hat{F}$ (respectively $\hat{F}_n$) be the piece-wise linear continuation of F (respectively F_n) as in Eq. (3), and denote by $\hat{f}$ (respectively $\hat{f}_n$) the PDF of $\hat{F}$ (respectively $\hat{F}_n$). Since $F_n \to F$, we also have $\hat{F}_n \to \hat{F}$ uniformly, as F_n is bounded and monotonic. Similarly, $\hat{f}_n \to \hat{f}$ almost everywhere. For each n, denote by d_n the solution of Eq. (4) with respect to $\hat{F}_n$.

We first show that

$$\lim_{n\to\infty} \left| \hat{F}(d_n) - \int_{d_n}^{\infty} \lceil x \rceil \, \mathrm{d}\hat{F}(x) \right| = 0. \tag{5}$$

Indeed, for every n,

$$\left| \hat{F}(d_n) - \int_{d_n}^{\infty} \lceil x \rceil \, \mathrm{d}\hat{F}(x) \right| \leq$$
$$\left| \hat{F}(d_n) - \hat{F}_n(d_n) \right| + \left| \hat{F}_n(d_n) - \int_{d_n}^{\infty} \lceil x \rceil \, \mathrm{d}\hat{F}_n(x) \right| + \left| \int_{d_n}^{\infty} \lceil x \rceil \, \mathrm{d}\hat{F}_n(x) - \int_{d_n}^{\infty} \lceil x \rceil \, \mathrm{d}\hat{F}(x) \right| \leq$$
$$\|\hat{F} - \hat{F}_n\|_{\infty} + 0 + \int_0^{\infty} (x+1)|\hat{f}(x) - \hat{f}_n(x)| \, \mathrm{d}x \xrightarrow{n\to\infty} 0.$$

Where the last term converges to 0 from Lebesgue's dominated convergence theorem.

Equation (5) implies that any partial limit of d_n solves Eq. (4) with respect to $\hat{F}$, and moreover that d_n is bounded as the set of solutions to this equation is bounded. Lastly, let d_∞ be a partial limit of d_n. We have

$$|\alpha(\hat{F}_n) - \alpha(\hat{F})| = |\hat{F}(d_\infty) - \hat{F}_n(d_n)| = |\hat{F}(d_\infty) - \hat{F}(d_n)| + |\hat{F}_n(d_n) - \hat{F}(d_n)| \xrightarrow{n\to\infty} 0,$$

where the first term converges to 0 from the continuity of $\hat{F}$, and the proof is complete.

5 Proof of Theorem 3

To prove Theorem 3 we utilize a characterization of minimal contagious from Morris (2000). An *acyclic orientation* of an (undirected) graph $G = (V, E)$ is a directed acyclic graph $D = (V, \vec{E})$ such that the mapping $\overrightarrow{vu} \mapsto vu$ is a bijection from $\vec{E}$ to E. We denote the set of all acyclic orientations[4] of G by $\mathrm{AO}(G)$. For every $D \in \mathrm{AO}(G)$ and every $\rho \in [0, 1]$, consider the set

$$\Lambda_\rho(D) := \{v \in V : (1 - \rho)d_v^{\mathrm{in}} \leq \rho d_v^{\mathrm{out}}\},$$

where d_v^{in} and d_v^{out} denote the in- and out- degrees of v in D respectively. The following two Lemmas provide a characterization of minimal contagious sets.[5]

Lemma 1. *For every network G every acyclic orientation D and every $\rho \in [0, 1]$, the set $\Lambda_\rho(D)$ is ρ-contagious.*

Proof. Consider the set of agents that are not activated by $\Lambda_\rho(D)$,

$$A := V \setminus \langle \Lambda_\rho(D), G \rangle_\rho.$$

If $A \neq \emptyset$, there exists a source to the sub-digraph induced by A, i.e. a vertex $v \in A$ with no incoming edges from A. Hence, all the incoming edges to v are from $\langle \Lambda_\rho(D), G \rangle_\rho$, and since $v \notin \Lambda_\rho(D)$, it has a ρ-majority of edges from $\langle \Lambda_\rho(D), G \rangle_\rho$. It implies that in the next step of the ρ-majority dynamics, v is activated, which is a contradiction.

We conclude that $A = \emptyset$ and $\Lambda_\rho(D)$ is contagious. □

Lemma 2. *For every network $G = (V, E)$, every $\rho \in [0, 1]$ and every ρ-contagious set $A \subset V$, there is an acyclic orientation D such that*

$$\Lambda_\rho(D) \subset A.$$

Proof. Let $A = A_0 \subset A_1 \subset \cdots$ be the sets of activated agents at times $0, 1, \ldots$ respectively. Let $v_1, v_2, \ldots$ be an ordering of the agents that agrees with the order of the sets $\{A_t\}$, that is, if $v_i \in A_t$ and $v_j \notin A_t$ then $i < j$. Define $D = (V, \vec{E})$ by

$$\vec{E} := \{\overrightarrow{v_i v_j} : v_i v_j \in E,\ i < j\}.$$

Let $v \notin A$. We must show that $v \notin \Lambda_\rho(D)$. Since A is contagious, there exists $t > 0$ such that $v \in A_t \setminus A_{t-1}$. All the edges between A_{t-1} to v are oriented in that direction and they form a ρ-majority of v's neighbors, therefore, $v \notin \Lambda_\rho(D)$. □

[4]The definition here refers to finite networks only. Morris (2000) considered infinite networks, in which case one should amend the definition by additionally requiring that the transitive closure of $\vec{E}$ is a well-founded partial order. Namely, there are no infinite sequences $v_1, v_2, \ldots$ such that $\overrightarrow{v_{k+1} v_k} \in \vec{E}$, for all $k \geq 1$.

[5]In the literature, there are models of contagion on directed networks. Here, the directions are only used as a tool for the proofs, and the activation process is still with respect to the undirected network.

Lemmas 1 and 2 immediately imply the following proposition.

Proposition 1. *For every $\rho \in [0,1]$, and every network G,*

$$h_\rho(G) = \min_{D \in \mathrm{AO}(G)} |\Lambda_\rho(D)|.$$

We are now ready to prove Theorem 3.

Proof of Theorem 3. Let $G = (V, E)$ be an (m_0, m)-attachment network of size n and $v_1, \ldots, v_n$ an ordering of the agents such that for every $l > m_0$ there are exactly m edges between v_l and $\{v_1, \ldots, v_{l-1}\}$.

Define

$$\vec{E} := \{\overrightarrow{v_i v_j} : v_i v_j \in E,\ i < j\},$$
$$\overleftarrow{E} := \{\overrightarrow{v_i v_j} : v_i v_j \in E,\ i > j\}.$$

Note that if $v_i \in \Lambda_\rho(V, \vec{E})$, then either $i \le m_0$, or $d_{v_i} \ge m/\rho$. Therefore,

$$|\Lambda_\rho(V, \vec{E})| \le m_0 + (1 - F(\lceil \tfrac{m}{\rho} - 1 \rceil))n.$$

Similarly, if $v_i \in \Lambda_\rho(V, \overleftarrow{E})$, then either $i \le m_0$, or $d_{v_i} \le \frac{m}{1-\rho}$. Therefore,

$$|\Lambda_\rho(V, \overleftarrow{E})| \le m_0 + F(\tfrac{m}{1-\rho})n.$$

By Lemma 1, both $\Lambda_\rho(V, \vec{E})$ and $\Lambda_\rho(V, \overleftarrow{E})$ are contagious, which concludes the proof. $\square$

6 Extensions

6.1 The Tightness of Theorem 1

In this section, we discuss the tightness of Theorem 1. We do so by referring to regular networks of varying degrees. A d-regular network is one in which the degree of all the agents is d.

Let's first spell out the lower bound obtained for such networks. As in the proof of Theorem 1, it is sufficient to consider $\rho = 0.5$. By (2), we obtain two different expressions for odd and even d.

$$h_{0.5}(G) \ge \begin{cases} \frac{n}{d+1} & d \text{ is odd,} \\ \frac{2n}{d+2} & d \text{ is even.} \end{cases}$$

In order for the seed to be able to activate any agent, its size must be greater than ρd. We thus have,

$$h_{0.5}(G) \ge \begin{cases} \frac{d+1}{2} & d \text{ is odd,} \\ \frac{d+2}{2} & d \text{ is even.} \end{cases}$$

Combining the two bounds we get the following proposition.

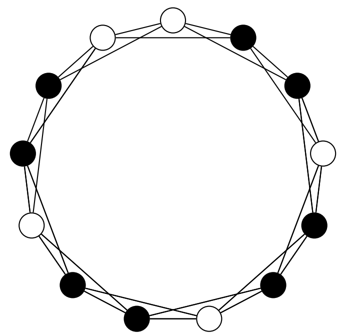

Figure 2: A Cayley graph with $n = 13$, $d = 4$ and a minimal contagious set (in white).

Proposition 2. *For any d-regular network G, and any $\rho \geq 0.5$, we have*

$$h_\rho(G) \geq \begin{cases} \max\{\frac{n}{d+1}, \frac{d+1}{2}\} & d \text{ is odd,} \\ \max\{\frac{2n}{d+2}, \frac{d+2}{2}\} & d \text{ is even.} \end{cases}$$

Note that Proposition 2 implies that $h_{0.5}(G) = \Omega(\sqrt{n})$, for any regular graph. The following examples prove the tightness of Proposition 2 up to a multiplicative factor of two. Specifically, for any $d \geq 2$, we construct a growing family of networks together with 0.5-contagious sets whose size is obtained by replacing the "maximum" in the expression of Proposition 2 with a "sum."

For d even (i.e., Eulerian graphs) consider the Cayley graph of $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ with generators $\{\pm 1, \ldots, \pm \frac{d}{2}\}$.

Consider the following set of size $\frac{2n}{d+2} + \frac{d}{2}$ (see Figure 2):

$$A_0 = \{0, \tfrac{d}{2}+1, 2(\tfrac{d}{2}+1), \ldots\} \cup \{-1, -2, \ldots, -\tfrac{d}{2}\}.$$

This set is contagious since, by induction, $t \in A_t$ for all $t = 0, 1, \ldots$.

For d odd, we construct an example for n divisible by $d+1$. We note that our construction can be modified slightly to fit any size n. Consider the Cayley graph of $\mathbb{Z}_2 \times \mathbb{Z}_{n/2}$ with generators

$$\left\{(1, x) : -\tfrac{d-1}{2} \leq x \leq \tfrac{d-1}{2}\right\}.$$

The following set of size $\frac{n}{d+1} + \frac{d-1}{2}$

$$A_0 = \{0\} \times \left(\left\{-1, \ldots, -\tfrac{d-1}{2}\right\} \cup \tfrac{d+1}{2}\mathbb{Z}_{n/2}\right)$$

will be shown to be contagious right below. The construction is depicted in Figure 3.

Since $A_0 \supset \{0\} \times \left(\left\{0, \ldots, -\frac{d-1}{2}\right\} \cup \left\{\frac{d+1}{2}\right\}\right)$,

$$A_1 \supset \{1\} \times \left\{1, \ldots, -\tfrac{d-1}{2}\right\},$$

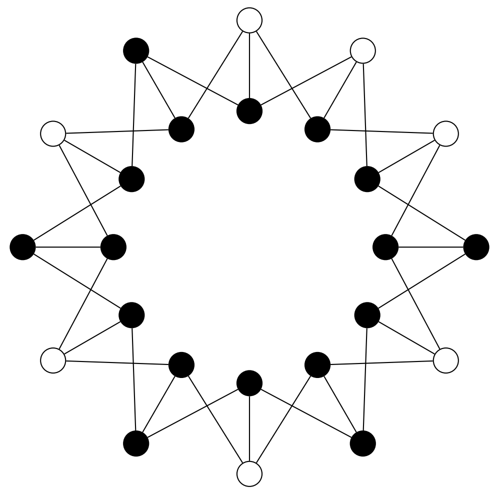

Figure 3: A bipartite graph with $n = 24$, $d = 3$, corresponding to $\mathbb{Z}_2 \times \mathbb{Z}_{12}$, and a minimal contagious set (in white).

and therefore,

$$A_2 \supset \{0\} \times \left(\left\{1, \ldots, -\tfrac{d-1}{2}\right\} \cup \left\{\tfrac{d+1}{2}\right\}\right).$$

It follows, by induction on t, that

$$A_{2t-1} \supset \{1\} \times \left\{t, \ldots, -\tfrac{d-1}{2}\right\}$$

and

$$A_{2t} \supset \{0\} \times \left(\left\{t, \ldots, -\tfrac{d-1}{2}\right\} \cup \left\{\left\lfloor \tfrac{2t}{d+1} + 1 \right\rfloor \tfrac{d+1}{2}\right\}\right),$$

for every $t \geq 1$.

6.2 The Tightness of Theorem 1 for Scale-Free Networks

We used a computer simulation to test the quality of our theoretical bound on the size of the minimal contagious seed for scale-free networks. In our simulation, we construct scale-free networks using the preferential attachment algorithm (Albert and Barabási, 2002), calculate the corresponding realized value of γ, and find (an approximation to) the minimal contagious seed. We then compare the size of this set to the predicted one according to Theorem 2, as can be seen in Figure 4.

Note that computing the minimal-size seed is NP-hard (Kempe et al., 2003), so we use the following algorithm to find an approximation:

Step 1: Start with a seed of 5 agents with the highest degree.

Step 2: Run the bootstrap percolation dynamics on the network until it reaches a steady state.

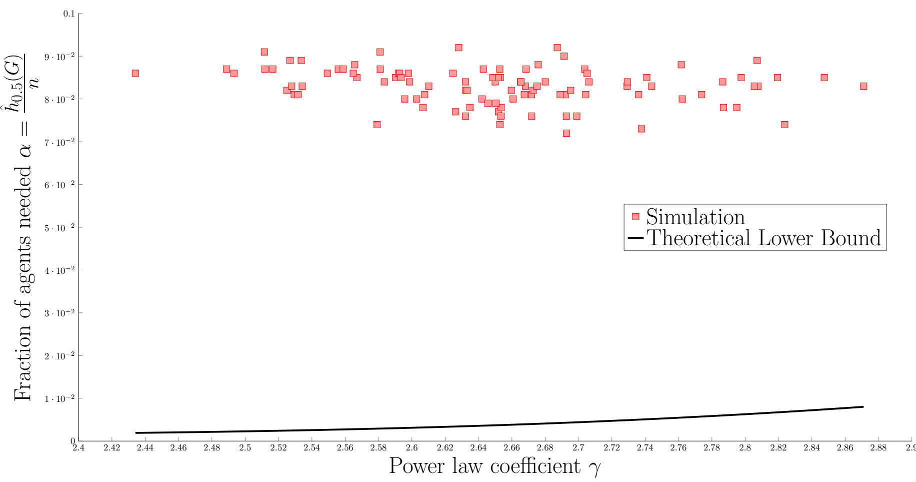

Figure 4: The theoretical lower bound based on Theorem 2 (black line) and simulated results for 100 networks of size $n = 1000$ (red squares).

Step 3: Add the inactive agent with the highest degree.

Step 4: Repeat steps $2 - 3$ iteratively, until in the entire population is activated.

Step 5: The seed set includes the initial 5 agents and all those who were added in the third step.

Naturally, this simple algorithm only provides an upper bound on $h_\rho(G)$, as the highest degree inactive agent might not be the best choice for the seed set (it might have too many wasteful connections, see Manshadi et al. (2020) for a similar discussion). We performed several tests with possible improvements, but they did not change the results significantly. We conclude that this algorithm gives a relatively tight upper bound on $h_\rho(G)$ for the simulation parameters, and we use it as an estimate of the real value of $h_\rho(G)$. To make the distinction, we use $\hat{h}_\rho(G)$ to refer to the results of the algorithm, $\hat{h}_\rho(G) > h_\rho(G)$.

We simulated 100 networks of size $n = 1000$ with $\rho = 0.5$, as can be seen in Figure 4. We observe that the theoretical lower bound is significantly lower than the approximated minimal size of the seed, a gap that cannot be explained by the fact that our algorithm does not find the seed of minimal size. We conclude that Theorem 2 is enough to establish the linear nature of the bound, but it overestimates the size of the discord set. The assumption that all the edges of the seed are in the discord set is too optimistic; in reality, the agents with the highest degree in scale-free networks are typically connected. Thus, the seed should be larger than what the theorem implies. More information regarding the structure of the network must be taken into account to obtain a tighter lower bound on $h_\rho(G)$, if needed. We leave such investigations to future research.

In addition, for small values of γ, the tail of the distribution is "fat" enough so that the limit $n \to \infty$ is not reached for $n = 1000$. In this case, computing the cut-off d^* under the assumption that $d_v < n$ adds an additional correction to the bound. This procedure is already implemented in Figure 4.

6.3 Discontinuity at $\rho = 0.5$

Theorems 1 and 2 provide linear lower bounds on $h_{0.5}(G)$ (and hence on $h_\rho(G)$, for $\rho \geq 0.5$ as well). In this section, we explain why such bounds do not exist for $\rho < 0.5$.

Let us denote

$$h_{0.5-}(G) := \lim_{\rho \nearrow 0.5} h_\rho(G).$$

There are some networks for which there is little or no difference between $h_{0.5}(G)$ and $h_{0.5-}(G)$ and other networks where the difference is dramatic. On one extreme, there is the complete graph K_n, for which $h_{0.5}(K_n) = \lceil n/2 \rceil$ and $h_{0.5-}(K_n) = \lfloor n/2 \rfloor$. Another example is any graph in which all the degrees are odd and then $h_{0.5}(G) = h_{0.5-}(G)$. On the other extreme, there is the line-shaped graph L_n, for which $h_{0.5}(L_n) = \lfloor n/2 \rfloor$, whereas, $h_{0.5-}(L_n) = 1$.

To illustrate a whole range of possible gaps between $h_{0.5}$ and $h_{0.5-}$ we consider the m-dimensional tori with $n = k^m$ vertices denoted C_k^m. Namely, the m-fold product of a k-cycle graph, $C_k^m = (V, E)$ where $V = \{1, \dots, k\}^m$, and

$$E = \left\{ \{x, y\} \in \binom{V}{2} : \exists i \in [m]\ x_i - y_i = \pm 1 \mod k,\ x_j = y_j\ \forall j \neq i \right\}.$$

The following proposition says that $h_{0.5-}(C_k^m) = \theta_m \left(n^{1-1/m} \right)$, whereas, $h_{0.5}(C_k^m) \geq \frac{n}{m+1}$, by Proposition 2.

Proposition 3. *For every $k, m \in \mathbb{N}$,*

$$\lfloor k/2 \rfloor^{m-1} \leq h_{0.5-}(C_k^m) \leq n - (k-1)^m \leq mn^{1-1/m}$$

Proof. To prove the upper bound consider the set $W = V \setminus \{2, \dots, k\}^m$. This set clearly has the required size. It remains to show that it is contagious w.r.t. to any $\rho < 0.5$.

Following Morris (2000) a set of vertices $B \subseteq V$ is called ρ-*cohesive* if for every $v \in B$, a proportion of at least ρ of its neighbors is inside B, namely, $|N_v \cap B| \geq \rho |N_v|$, and a set is ρ-contagious if and only if it intersects all the nonempty $(1-\rho)$-cohesive sets.

Let $\rho < 0.5$ and suppose for the sake of contradiction, that there exists a nonempty $(1-\rho)$-cohesive set B that does not intersect W, namely, $\emptyset \neq B \subset \{2, \dots, k\}^m$. Let $x = (x_1, \dots, x_m) \in \arg\min_{y \in B} \sum_{i=1}^m y_i$. Since $\deg(x) = 2m$, and B is $(1-\rho)$-cohesive, there must be i such that $x - e_i \in B$ (as well as $x + e_i \mod k$), where $e_1, \dots, e_m$ is the standard basis of $\mathbb{R}^m$. Alas, $x - e_i$ contradicts the minimality of x.

To prove the lower bound it is sufficient to find $\lfloor k/2 \rfloor^{m-1}$ disjoint $(1-\rho)$-cohesive sets. Indeed, for any $i_1, \dots, i_{m-1} \in \{1, \dots, \lfloor k/2 \rfloor\}$, the set

$$\{2i_1 - 1, 2i_1\} \times \cdots \times \{2i_{m-1} - 1, 2i_{m-1}\} \times \{1, \cdots, k\}$$

is $(1-\rho)$-cohesive, and these sets are disjoint. □

7 Discussion

We study innovation diffusion via the bootstrap percolation model in scale-free networks and show that under the optimal seeding strategy, the seed size is linear in the size of the network. We generalize this statement to a vast variety of networks of increasing size, provided that their degree distribution converges, the average degree converges ($\mathbb{W}_1$-converging networks, Definition 1) and the average is finite (Theorem 2). Our research provides insights into the dynamics of innovation diffusion in scale-free networks and offers practical implications for industries and policymakers aiming to promote technology adoption and diffusion. By understanding the minimal seed size required for successful innovation adoption, targeted seeding strategies can be developed to leverage early adopters and accelerate the diffusion process.

Our work is different from previous studies regarding percolation on networks. The main difference is that we study it asymptotically, as the network size increases while the degree distribution converges to some limit distribution, whereas the literature mainly considers networks of fixed size. Moreover, we do not assume that the degree distribution is bounded. Hence, as n increases, the maximal degree in the network can also increase to infinity. This is in contrast with previous studies of the size of contagious sets that considered sequences of networks with growing sizes (or simply infinite networks) but under the assumption that the degree is uniformly bounded (Candogan, 2022; Manshadi et al., 2020; Morris, 2000). In addition, we study the scale-free networks with a relative activation threshold, which departs from the common models both in the network structure and the activation threshold.

In this paper, we show that the seed size is linear in the size of the population. Both the simulated threshold (10%) and the theoretical one (1%) are high and require a significant effort in terms of seeding to reach full adoption. We conclude that from a marketing perspective, it is not feasible to require a full adoption of the new technology and a modest objective is needed.

Our work regarding scale-free networks focuses on one aspect of this type of network – the converging degree distribution. We do not take into account other properties of the networks, such as the inner connectivity of high-degree vertices. Taking this and other properties into account can close the gap between the theoretical and simulated results, both by improving the theoretical bound and by improving algorithms for finding the optimal seed (or its approximation, see also Akbarpour et al. (2020)). The latter, along with understanding the path the percolation process makes inside the network, is essential to understand what are the optimal seeding strategy and the expected percolation time. This is left for further research.

Finally, we note that if the goal is to activate only a portion of the network, our results scale accordingly: to activate half of the network we need (order of) $0.5h_\rho(G)$ seeds. This has no impact on our main result (Theorem 2) which concerns with the asymptotic nature of the bound as the size of the network increases. We conclude that if we want to activate a linear fraction of the network, the seed set must be itself a linear fraction of the network, in contrast to other models, such as Amini et al. (2013).

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